J. Fluid Mech. (1965), *vol.* 21, *part* 2, *pp.* 83–95
Printed in Great Britain

Horizontal mixing in the sea due to a shearing current

By K. F. BOWDEN
Oceanography Department, University of Liverpool

(Received 15 June 1964)

In addition to the random processes described as horizontal turbulence, there are certain more regular processes which may contribute to horizontal mixing. One of these occurs in a shearing current, where the vertical gradient of velocity combined with vertical turbulent mixing leads to an effective diffusion in the horizontal direction. It is shown that this effect occurs in an alternating flow, such as a tidal current, as well as in a steady flow. In estuaries and coastal waters horizontal mixing by the shear effect may be associated with tidal currents, density currents or wind-driven currents. In each case the effective coefficient of horizontal diffusion, K_x or K_y, is inversely proportional to the coefficient of vertical eddy diffusion K_z. The occurrence of a stable gradient of density therefore increases the effective horizontal mixing very considerably. Results obtained from observations in the Mersey estuary and Irish Sea are compared with theoretical estimates of these effects.

1. Introduction

The horizontal mixing of two water masses or the dispersion of a patch of dye, or other contaminant, is usually regarded as being due to horizontal turbulence and is treated by the methods used for turbulent diffusion. This treatment implies a degree of randomness in the water movements which produce the mixing. The mixing, however, may be the result of patterns of flow set up by tidal currents, wind drift or waves, and these have a certain regularity, related to the cause producing them. It is unlikely that the same method of statistical treatment or the application of a general theory of turbulence, will apply to all the processes which occur. In some cases it may be more useful to consider the actual pattern of flow which produces the mixing.

One example of this is the process which has been called the 'shear effect' (Bowles, Burns, Hudswell & Whipple 1958), the idea of which was first used by Taylor (1953, 1954) for the case of the flow of liquid through a pipe. A similar treatment was given by Elder (1959) for turbulent flow in an open channel. The basic result is that an effective longitudinal dispersion is produced by the combination of a transverse gradient of velocity with transverse turbulent mixing. In the case of pipe flow, the transverse variation of velocity is due to friction at the wall, but in general a velocity profile due to any cause may give rise to the shear effect.

2. Steady-state theory

Horizontal flow in a channel of depth h will be considered. Let rectangular axes be taken with x measured along the bed of the channel, y across it and z vertically upwards. The mean velocity at any depth is assumed to be in the x-direction and is denoted by u. Let S be the concentration of salt or of any other indicator substance in the water. It is assumed that variations in conditions across the channel are negligible. Let a bar over any quantity denote a mean value with respect to depth, e.g.

$$\bar{u} = \frac{1}{h}\int_0^h u\,dz.$$

Then the equation governing the distribution of the quantity S may be written

$$\frac{\partial S}{\partial t}+u\frac{\partial S}{\partial x} = \frac{\partial}{\partial z}\left(K_z\frac{\partial S}{\partial z}\right), \tag{1}$$

where K_z is the coefficient of eddy diffusion in the vertical direction. In the treatments of Taylor and of Elder, a transformation is made to axes moving with the mean flow, i.e. with velocity $\bar{u}$. A solution is then found for the case in which $\partial S/\partial t = 0$ relative to the moving axes and $\partial S/\partial x$ is independent of z. The vertical distribution of S is given by

$$\frac{\partial}{\partial z}\left(K_z\frac{\partial S}{\partial z}\right) = u_1\frac{\partial S}{\partial x}, \tag{2}$$

where $u_1 = u-\bar{u}$. This solution is appropriate to the case of a patch of dye, for example, which is released along a vertical line in the fluid and then moves with it, at the same time becoming dispersed.

A similar solution applies to the concentration along a fixed vertical in the channel, if the deviation of S from the depth-mean value is considered. Thus by integrating (1) from $z = 0$ to $z = h$ and expressing the condition that the flux of S through each boundary is zero,

$$\frac{\partial \bar{S}}{\partial t}+\bar{u}\frac{\partial S}{\partial x} = 0. \tag{3}$$

From (1) and (3),

$$\frac{\partial S_1}{\partial t}+u_1\frac{\partial S}{\partial x} = \frac{\partial}{\partial z}\left(K_z\frac{\partial S}{\partial z}\right), \tag{4}$$

where $u_1 = u-\bar{u}$, and $S_1 = S-\bar{S}$. If $\partial S_1/\partial t = 0$, equation (4) reduces to (2).

Let the velocity profile be given by

$$u = Uf(\zeta), \tag{5}$$

where U is the surface value of u and $f(\zeta)$ is some function of ζ, where $\zeta = z/h$. Similarly,

$$u_1 = Uf_1(\zeta), \tag{6}$$

where

$$f_1(\zeta) = f(\zeta)-\overline{f(\zeta)}.$$

Let the coefficient of vertical eddy-diffusion be any prescribed function of ζ, i.e.

$$K_z = Kg(\zeta), \tag{7}$$

where K is the maximum value of K_z in the vertical section and $0 < g(\zeta) < 1$.

The horizontal gradient $\partial S/\partial x$ is assumed to be independent of z and may be written $\partial \bar{S}/\partial x$. By integrating (2) it is found that

$$S = \bar{S} + \frac{Uh^2}{K}\frac{\partial \bar{S}}{\partial x}[F(\zeta) - \overline{F(\zeta)}], \tag{8}$$

where
$$F(\zeta) = \int \left\{ [1/g(\zeta)] \int f_1(\zeta)\, d\zeta \right\} d\zeta.$$

The mean rate of transport of the property S across a vertical section due to the velocity u_1 is given by

$$\overline{u_1 S_1} = \frac{U^2h^2}{K}\frac{\partial \bar{S}}{\partial x}\overline{f_1(\zeta)\,F(\zeta)}. \tag{9}$$

The mean total transport across a fixed section is $\bar{u}\bar{S} + \overline{u_1 S_1}$, where the term $\bar{u}\bar{S}$ represents advection by the mean flow. If no account had been taken of the variation of u and S with depth, the term $\overline{u_1 S_1}$ would have been attributed to horizontal eddy diffusion and it would have been written as

$$\overline{u_1 S_1} = -K_x\, \partial \bar{S}/\partial x, \tag{10}$$

where K_x is the coefficient of eddy diffusion in the x-direction. Equating equations (9) and (10),

$$K_x = -(U^2h^2/K)\overline{f_1(\zeta)\,F(\zeta)}. \tag{11}$$

Hence the effective value of the longitudinal coefficient of diffusion, K_x, may be expressed in terms of the velocity profile and functions derived from it. A significant feature of this equation is that K_x is *inversely* proportional to the magnitude of the coefficient of vertical eddy diffusion, other factors being equal.

3. Shear effect in alternating flow

In many estuaries and coastal regions the predominant water movements are the tidal currents, oscillating with a period of half a lunar day. In considering the dispersion of a patch of pollutant over a short length of time, e.g. 1 h, it may be sufficient to regard the conditions as quasi-stationary and apply the steady-state theory, using a mean value for u over this period. In studying conditions over a longer period, however, the variations with time must be taken into account. It will be shown that, considering the mean conditions over one or more complete tidal periods, horizontal diffusion by the shear effect can still occur.

The current is assumed to be in the x-direction and to consist of a single harmonic constituent, i.e.

$$u = A(z)\cos\sigma t + B(z)\sin\sigma t, \tag{12}$$

where $A(z)$ and $B(z)$ are functions of z and σ is the angular frequency of the tidal constituent.

The equation for S_1 is, from (4),

$$\frac{\partial}{\partial z}\left(K_z\frac{\partial S_1}{\partial z}\right) = \frac{\partial S_1}{\partial t} + u_1\frac{\partial \bar{S}}{\partial x}, \tag{13}$$

where
$$u_1 = A_1\cos\sigma t + B_1\sin\sigma t \tag{14}$$

and $A_1 = A(z) - \overline{A(z)}$, etc. Assume a solution for S_1 in the form

$$S_1 = P\cos\sigma t + Q\sin\sigma t, \tag{15}$$

where P and Q are functions of z. If it is assumed that K_z does not vary with time during the tidal period, it follows from (13) and (15) that

$$\frac{\partial}{\partial z}\left(K_z\frac{\partial P}{\partial z}\right) = \sigma Q + A_1\frac{\partial\bar{S}}{\partial x}, \tag{16}$$

$$\frac{\partial}{\partial z}\left(K_z\frac{\partial Q}{\partial z}\right) = -\sigma P + B_1\frac{\partial\bar{S}}{\partial x}. \tag{17}$$

In many cases the variation of phase of the tidal current with depth is comparatively small. In such a case the time origin may be chosen so that $\bar{B} = 0$ and $B \ll A$ for all values of z. If it is assumed that in this case Q is small compared with P, a first approximation to P may be found by putting $Q = 0$ in equation (16). Using this value of P, Q may be found from (17) and its value used in (16) to obtain a second approximation to P.

It is in fact unrealistic to regard K_z as remaining constant during the tidal period, since it will vary with the intensity of turbulence present. A more reasonable assumption is to take K_z as proportional to $|\bar{u}|$, the magnitude of the depth-mean current at any time. By choosing the time origin so that $\bar{B} = 0$, K_z may be represented by

$$K_z = k_z|\cos\sigma t|, \tag{18}$$

where k_z is a function of z only.

On substituting from (18) in (13), terms $|\cos\sigma t|\cos\sigma t$ and $|\cos\sigma t|\sin\sigma t$ arise and these may be expanded as Fourier series. The solution (15) must then be taken in the form

$$S_1 = P_1\cos\sigma t + P_2\cos 2\sigma t + \ldots + Q_1\sin\sigma t + Q_2\sin 2\sigma t + \ldots. \tag{19}$$

For the functions P_1 and Q_1,

$$\frac{\partial}{\partial z}\left(k_z\frac{\partial P_1}{\partial z}\right) = \frac{3\pi}{8}\left(\sigma Q_1 + A_1\frac{\partial\bar{S}}{\partial x}\right), \tag{20}$$

$$\frac{\partial}{\partial z}\left(k_z\frac{\partial Q_1}{\partial z}\right) = \frac{3\pi}{4}\left(-\sigma P_1 + B_1\frac{\partial\bar{S}}{\partial x}\right), \tag{21}$$

analogous to (16) and (17). Thus the first-order terms differ from those in the case of a constant K_z only by a numerical factor.

Taking u_1 from (14) and S_1 from (15), the depth-mean transport $\overline{u_1S_1}$ may be determined and this will vary during the tidal period. Over a complete period T, however,

$$[u_1S_1]_T = \tfrac{1}{2}(\overline{A_1P} + \overline{B_1Q}), \tag{22}$$

where $[\ ]_T$ denotes a mean value over the tidal period.

As an example of the use of this equation, consider the approximate solution for the case of a constant K_z and $B \ll A$. The solution of (16) neglecting σQ compared with $A_1\,\partial\bar{S}/\partial x$, is

$$P = R_1\,\partial\bar{S}/\partial x, \quad \text{where} \quad R_1(z) = \int(1/K_z)\left(\int A_1\,dz\right)dz. \tag{23}$$

Then from (17)

$$Q = R_2\,\partial\bar{S}/\partial x, \quad \text{where} \quad R_2(z) = \int (1/K_z)\left\{\int (-\sigma R_1 + B_1)\,dz\right\} dz. \tag{24}$$

Then
$$[\overline{u_1 S_1}]_T = \tfrac{1}{2}(\partial\bar{S}/\partial x)\,(\overline{A_1 R_1} + \overline{B_1 R_2}), \tag{25}$$

and the effective value of K_x is given by

$$K_x = -\tfrac{1}{2}(\overline{A_1 R_1} + \overline{B_1 R_2}). \tag{26}$$

If A_1 is written as $A_1 = Uf_1(\zeta)$ and K_z as $K_z = Kg(\zeta)$, as in the steady-state case (equations (6) and (7)),

$$R_1(z) = (Uh^2/K)\,[F(\zeta) - \overline{F(\zeta)}], \tag{27}$$

where $F(\zeta)$ is as in equation (8). Hence the term $-\overline{A_1 R_1}$ in (26) is given by

$$-A_1 R_1 = -(U^2h^2/K)\overline{f_1(\zeta)\,F(\zeta)}, \tag{28}$$

which is equivalent to K_x for the steady-state (equation (11)). Thus if the term $\overline{B_1 R_2}$ is small compared with $\overline{A_1 R_1}$ in (26),

$$K_x \text{ (alternating flow)} \approx \tfrac{1}{2}K_x \text{ (steady flow)}.$$

This approximate relationship is valid provided:

(i) the surface amplitude of the tidal current is equal to the surface velocity of the steady flow;

(ii) the variation of the amplitude with depth is similar to the steady-state velocity profile;

(iii) the variation in phase of the tidal current with depth is small;

(iv) the vertical eddy diffusivity is the same and varies in the same way with depth;

(v) the depth of water is the same.

4. Some particular velocity profiles and corresponding values of K_x

The salinity profile and the effective value of K_x have been computed for several typical velocity profiles, making certain assumptions about K_z. These cases are

(1) The velocity profile is taken as logarithmic over the whole depth, as in Elder's (1959) application of Taylor's theory to flow in an open channel, i.e.

$$u = u_* k_0^{-1} \ln(\zeta/\zeta_0), \tag{29}$$

where $\zeta_0 = z_0/h$, z_0 is the roughness length and k_0 is von Kármán's constant, equal to 0·41. Neutral stability is assumed so that K_z may be taken as equal to the vertical eddy viscosity N_z. This implies a parabolic variation of K_z with depth, i.e.

$$K_z = k_0 u_* h\zeta(1-\zeta). \tag{30}$$

(2) The velocity is taken as varying logarithmically from the bottom up to $\zeta = \alpha$ and according to a parabolic law above it. This form of profile agrees with some observations of tidal currents rather better than the purely logarithmic one.

In neutral conditions and with $K_z = N_z$ again, this corresponds to K_z increasing linearly up to $\zeta = \alpha$ and then remaining constant. From observations of tidal currents in the Red Wharf Bay area of the Irish Sea (Bowden & Fairbairn 1952), the best fit of such a profile was found by taking $\alpha = 0{\cdot}14$. This corresponds to the observed value of the coefficient of bottom friction, $k = 2\times 10^{-3}$, and a ratio of mean amplitude to surface amplitude of 0·87. Then for $0 < \zeta < 0{\cdot}14$,

$$u = (u_*/k_0)\,[\ln(\zeta/\zeta_0) - \zeta + \zeta_0], \tag{31}$$

$$K_z = k_0 u_* h\zeta. \tag{32}$$

For $0{\cdot}14 < \zeta < 1$,

$$u = u_* k_0^{-1}[\alpha^{-1}(\zeta - \tfrac{1}{2}\zeta^2) + \ln(\alpha/\zeta_0) - \tfrac{1}{2}\alpha - 1], \tag{33}$$

$$K_z = \alpha k_0 u_* h, \tag{34}$$

with $\alpha = 0{\cdot}14$.

(3) From a large number of measurements of tidal currents in Dutch estuaries and in the Straits of Dover, Van Veen (1938) found that near the time of maximum flood or ebb flow the velocity profile was represented very closely by a power law of the form

$$u = U\zeta^{\beta}, \tag{35}$$

where U is the surface velocity and β a constant. For the Straits of Dover the best fit with the observations was obtained with $\beta = 1/5{\cdot}2$. More recent measurements of currents in the same area by Cartwright (1961) have confirmed the validity of this equation. In steady flow, it may be shown that the vertical eddy viscosity, and hence the vertical eddy diffusivity under the conditions assumed, is then given by

$$K_z = K_0\zeta^{1-\beta}(1-\zeta), \tag{36}$$

where $K_0 = k^{\frac{1}{2}}u_* h/\beta(1+\beta)$, and k is a friction coefficient, relating the bottom stress to the mean velocity, i.e.

$$\tau_0 = \rho u_*^2 = k\rho\bar{u}^2. \tag{37}$$

(4) The same velocity profile is assumed as in (2), but K_z is taken as constant throughout the depth. This simplification is made in order to facilitate comparison with the profiles in (5) and (6).

(5) The velocity is given by

$$u = \tfrac{1}{2}U(3\zeta^2 - 1). \tag{38}$$

This represents an approximation to a density current profile and is purely empirical, based on the type of profile which has been observed. K_z is taken as constant.

(6) This example is based on Ekman's theory of wind drift in shallow water (Ekman 1905; Defant 1961). The two cases of $h = 0{\cdot}25D$ and $h = 1{\cdot}25D$, where D is the depth of frictional influence, have been considered. In these two cases, particularly the latter, the current direction varies appreciably with depth and K_x and K_y have been computed, parallel and perpendicular respectively, to the direction of the surface current. K_z is taken as constant.

The values of K_x, calculated for these particular cases by the method of § 2, are shown in table 1. The value for Case 1, $K_x = 5{\cdot}9u_* h$, is in agreement with that in Elder's paper. The second value, $K_x = 14{\cdot}0u_* h$, corresponds to the composite velocity profile, changing from logarithmic to parabolic at $\zeta = 0{\cdot}14$. The higher value of the coefficient is due to the greater variation of velocity with depth and the lower value of K_z in the middle part of the depth. The increase in K_x by a factor of $2\frac{1}{2}$ illustrates the large effect of a comparatively small change in the form assumed for the velocity profile. If the bottom friction τ_0 is related to the mean velocity by equation (37) and the coefficient k is taken as 2×10^{-3}, $14{\cdot}0u_* h$ becomes $0{\cdot}625\bar{u}h$. Assuming K_z to be constant down to the bottom, as in Case 4 (*a*), makes comparatively little difference to K_x.

Conditions	Equation for K_x: In terms of u_*	Equation for K_x: In terms of $\bar{u}$ ($k = 0{\cdot}002$)
1. Logarithmic velocity profile (Elder)	$5{\cdot}9\, u_* h$	$0{\cdot}26\, \bar{u}h$
2. Log + parabolic profile $\alpha = 0{\cdot}14$, $K_z = N_z$	$14{\cdot}0\, u_* h$	$0{\cdot}625\, \bar{u}h$
3. Van Veen's profile $1/\beta = 5{\cdot}2$, $K_z = N_z$	$24{\cdot}9\, u_* h$	$1{\cdot}11\bar{u}h$
4. Log + parabolic profile (*a*) $K_z = \text{const.} = \alpha k_0 u_* h$	$13{\cdot}5u_* h$	$0{\cdot}60\bar{u}h$
	In terms of U	
(*b*) $K_z = \text{const.}$ (general case)	$1{\cdot}25\Phi$ where $\Phi = 10^{-3}(U^2h^2/K_z)$	
5. Density current (parabolic) $K_z = \text{const.}$	$19{\cdot}05\Phi$	
6. Wind drift. $K_z = \text{const.}$ (*a*) $h = 0{\cdot}25D$	$K_x = 8{\cdot}3\Phi$, $K_y = 0{\cdot}02\Phi$	
(*b*) $h = 1{\cdot}25D$	$K_x = 5{\cdot}3\Phi$, $K_y = 1{\cdot}1\Phi$	

TABLE 1. Effective values of K_x for various velocity profiles.

In case 4 (*b*), the same velocity profile is assumed and K_z is taken to be constant, but no assumption is made about its magnitude. K_x is expressed in the form given in order to be able to compare its value in a steady gradient current, or in a tidal current under the conditions specified at the end of § 3, with that in other types of current for the same surface velocity U and the same constant value of K_z. The use of Van Veen's formula (Case 3), with $\beta = 1/5{\cdot}2$ and $k = 2 \times 10^{-3}$, leads to a value of K_x nearly twice that corresponding to the logarithmic + parabolic profile and four times Elder's value.

The value of K_x appears much greater for a density current, as in Case 5, than for a gradient current of the same surface velocity (Case 4 (*b*)) because in the latter case the variation in velocity with depth is small over the greater part of the depth, whereas in a density current it is large, in fact becoming negative in the lower part. In the two cases of wind drift 6(*a*) and 6 (*b*), K_x is 4 to 7 times as great as in a gradient (or tidal) current of the same surface velocity, again due to the greater variation of velocity with depth, particularly near the surface. In the case of $h = 0{\cdot}25D$, the variation of direction with depth is so small that the computed value of K_y is negligible, compared with K_x. In practice K_y would then be governed by other processes, such as random variations in the wind direction. For $h = 1{\cdot}25D$, however, the computed value of K_y reaches one-fifth that of K_x.

At this point the influence of a stable density gradient may be emphasized. The general effect would be to reduce the value of K_z and in each type of flow the effective value of K_x would be increased accordingly.

5. Application to mixing in the sea

(*a*) *Spreading of a dye patch*

Experiments on the spreading of a patch of dye in a tidal current in an area of the English Channel were described by Bowles *et al.* (1958). From the observations they found an average value of K_x of $1{\cdot}8 \times 10^5\,\text{cm}^2/\text{sec}$, which corresponds to $K_x = 1{\cdot}21\bar{u}h$. In these experiments the density of the water was practically uniform with depth. The dye, released near the bottom, spread through a vertical column within about half an hour and its horizontal dispersion was followed for 7 h. The coefficient K_y, corresponding to dispersion in a direction transverse to the current, was about one-tenth of K_x. By a method similar to that of Elder, Bowles *et al.* derived theoretical estimates of K_z corresponding to $0{\cdot}15\bar{u}h$ and $0{\cdot}36\bar{u}h$, depending on alternative assumptions about the variation of the vertical eddy diffusivity with depth. From table 1 it is seen that, with the friction coefficient $k = 2 \times 10^{-3}$, Elder's equation corresponds to $0{\cdot}26\bar{u}h$ and the logarithmic + parabolic velocity profile result to $0{\cdot}625\bar{u}h$. The use of Van Veen's equation leads to $K_x = 1{\cdot}11\bar{u}h$, which gives the nearest approach to the experimental result.

A number of other investigations on the spreading of dye patches have been reviewed by Okubo (1962) and Bowden (1964). The influence of a wind in elongating the patch in the direction of the wind has been reported by several authors. Thus Ichiye (1962), from observations off Panama City, Florida, in varying conditions of wind and current, found that a noticeable elongation occurred when the wind speed exceeded 3 m/sec or the surface current exceeded 10 cm/sec. Such an elongation can clearly be explained as a consequence of the current shear in the water. The effect of varying current patterns on a diffusing plume of dye has been described by Csanady (1963), from observations in Lake Huron.

Various treatments of horizontal diffusion in the sea have been given in which the coefficient of diffusion is taken to be a function of the scale of the phenomena only. In the theory of Joseph & Sendner (1958), for example, the coefficient K_r in a radial direction is taken as $K_r = Pr$, where P is a 'diffusion velocity' and r is the radial distance from the origin reached by a diffusing particle. These authors found agreement with a number of observations, on a comparatively large scale, by giving P a constant value of 1 cm/sec. In this and similar methods, no allowance is made explicitly for factors such as wind speed and the velocity of tidal currents. The explanation for the fact that satisfactory agreement with observations can be obtained in this way is probably that, over comparatively large distances and long intervals of time, a variety of wind and current conditions is encountered and the average effect does not differ greatly from one case to another. In considering diffusion in a smaller area and over a shorter time scale, the observations referred to earlier have shown a dependence on the particular physical conditions.

(*b*) *Distribution of salinity*

A special case of horizontal diffusion is the spreading of fresh water in an estuary and its mixing with the sea water. Since the salinity can be determined with a high degree of accuracy, the fresh water provides a useful indicator of the mixing processes. The situation is complicated, however, since a change of salinity implies a change of density and the water can no longer be homogeneous. A vertical gradient of density alters the stability and affects the vertical eddy diffusivity, while a horizontal density gradient can generate a density current, altering the pattern of flow. However, if the density changes are comparatively small, one might expect the analysis for a 'neutral' indicator to hold approximately for salinity.

A common problem involves the distribution of salinity in a tidal estuary, or coastal area, when conditions averaged over one or more complete tidal periods are assumed to be steady. The shear effect due to the tidal current alone, assuming the departure from neutral stability to be insignificant, would contribute an effective K_x which can be computed by the method of §§ 3 and 4 above. Calculations made in this way for the estuaries of the Severn, Thames and Mersey (Bowden 1963), give values of K_x which are smaller than the observed values by a factor varying from 10 to 100. If the stability effect were sufficient to reduce the vertical eddy diffusivity by a factor of this order, the tidal currents could produce an effective K_x of the right magnitude. At the same time, the shear effect arising from the density current flow produced in the estuary would also add to the effective horizontal diffusion.

6. Direct determination of salt transport due to the shear effect

The rate of transport of salt by the shear effect may be determined directly by measuring the current velocity at various depths and the corresponding salinity values and taking the deviation of each from the depth-mean. The product $u_1 S_1$ at each depth is formed and then the depth-mean $\overline{u_1 S_1}$. By repeating the measurements at intervals of, say, 1 h throughout the tidal period, the average transport over the period may be determined. This may be compared with the horizontal gradient of salinity to give the effective value of K_x.

Observations which could be treated in this way were made at positions in the Mersey estuary (Bowden 1963). The average of a number of determinations gave $K_x = 1{\cdot}7 \times 10^6\,\mathrm{cm^2/sec}$. An independent estimate of K_x was made by assuming that the salinity distribution, averaged over a tidal period, was in a steady state and that the seaward advection of salt by the river flow was balanced by the landward turbulent transport. K_x is then given by the equation

$$K_x = R\bar{S}/A(\partial\bar{S}/\partial x), \tag{39}$$

where R is the rate of river discharge and $\bar{S}$ is the mean salinity over a cross-section of area A. This method includes the transport due to genuine horizontal turbulence or to differences in salinity between the flood and ebb flows, as well as the shear effect. As expected, the values of K_x estimated in this way were, in general, greater than those found by the previous method, the average values of K_x being $2{\cdot}65 \times 10^6\,\mathrm{cm^2/sec}$.

If the horizontal transport were due solely to the tidal currents and the water were homogeneous, the effective value of K_x should be given approximately by

$$K_x = 0{\cdot}31\bar{u}h, \tag{40}$$

where $\bar{u}$ is the depth-mean amplitude of the tidal current. This uses the result from § 3 that K_x for an alternating current is approximately half that for a steady current under similar conditions, and the result given in table 1 for a logarithmic + parabolic velocity profile. For the Mersey, $\bar{u} = 150\,\mathrm{cm/sec}$, $h = 20\,\mathrm{m}$ and hence $K_x = 0{\cdot}93 \times 10^5\,\mathrm{cm^2/sec}$. The observed values are therefore 18 times those computed on this assumption. To reconcile the two one would have to assume a reduction in the coefficient of vertical eddy diffusion K_z by this factor. It is shown below that such a reduction would be consistent with the degree of stability present.

Similar observations made at a station in Liverpool Bay and at two stations off the Cumberland coast gave effective values of K_x and K_y which were of the order of $5 \times 10^5\,\mathrm{cm^2/sec}$. These were also considerably higher than could be accounted for by the shear effect if conditions of neutral stability applied.

7. Determination of K_z

If it is assumed that the vertical distribution of salinity, averaged over a tidal period, represents a balance between horizontal advection and vertical eddy diffusion, it is possible to calculate an effective value of K_z from the observations. The relevant equation, derived from (4), is

$$\frac{\partial}{\partial z}\left(K_z \frac{\partial S_T}{\partial z}\right) = \frac{1}{T}\int_0^T \left(u\frac{\partial S}{\partial x} - \overline{u\frac{\partial S}{\partial x}}\right) dt, \tag{41}$$

where S_T is the salinity, at depth z, averaged over the tidal period T and the over bar denotes a mean value with respect to depth. Some results obtained by this method are shown in table 2. K_z is given for various values of z/h, the relative distance from the bottom. The local value of the Richardson number Ri, is given by

$$\mathrm{Ri} = \frac{g}{\rho}\frac{\partial \rho}{\partial z}\bigg/\left(\frac{\partial u}{\partial z}\right)^2. \tag{42}$$

The value of $(\partial u/\partial z)^2$ was computed for the tidal current at each hour and then a mean taken over the tidal period. The general pattern of the K_z values is the same in each case: K_z is lowest near the surface, reaches a maximum near mid-depth, and then decreases towards the bottom, while still remaining higher than near the surface. The maximum values of K_z are an order of magnitude smaller than one would expect in neutral stability if K_z equalled N_z, the coefficient of vertical eddy viscosity.

Position	z/h	K_z (cm²/sec)	Ri
Mersey estuary	0·9	6	0·6
(mean of 7 stations)	0·7	17	0·7
	0·5	32	0·5
	0·3	26	0·3
	0·1	16	0·13
Liverpool Bar	0·9	17	0·24
	0·7	36	0·17
	0·5	36	0·23
	0·3	39	0·17
	0·1	20	0·06
Off Cumberland	0·9	2	0·6
(mean of 2 stations)	0·7	3·5	2·0
	0·5	4·5	1·3
	0·3	10	1·7
	0·1	7·5	0·25

TABLE 2. Values of K_z and Ri, determined from mean conditions over a tidal period.

The above procedure for estimating Ri was adopted because it is the tidal currents which are mainly responsible for the turbulent mixing. If, however, one attempts to compute Ri for individual hours of the tidal cycle, the results show a very wide scatter, due to the changing profile of the current, which at times reached a maximum or minimum value at some intermediate depth. The density always increased with depth but the gradient varied, being greatest near low water and least at half flood. As an indication of the overall properties of the flow in the three localities considered, table 3 gives the values of h, U and $\Delta\sigma_t$, where h is the mean depth of water, U is the surface amplitude of the tidal current and $\Delta\sigma_t$ represents the difference between the surface and bottom values of the density, averaged over a tidal period ($\rho = 1 + 10^{-3}\sigma_t$).

Position	h (m)	U (cm/sec)	$\Delta\sigma_t$
Mersey estuary	19·5	141	0·87
Liverpool Bar	16·0	59	0·19
Off Cumberland	24·5	40	0·35

TABLE 3. Values of h, U and $\Delta\sigma_t$.

8. Interaction of tidal current and density current

These examples, especially the Mersey estuary, represent an interaction between density-current and tidal-current effects. The method of determining the salt transport by computing the $\overline{u_1 S_1}$ terms includes both effects, and in fact it is not practicable to separate the two, as was done in the simple examples considered theoretically. The conditions change within the tidal period, advective effects predominating at one time and vertical mixing effects at another. The value of K_z no doubt varies during the tidal period and the method described above gives only an effective value for the mean conditions. Similar changes, although on a less pronounced scale, occur at the stations further from the coast.

9. Range of usefulness of the concept of an effective K_x

In the case of dispersion in flow along a pipe, for which Taylor originally developed the theory of the shear effect, the dye was, from a practical point of view, distributed uniformly across a section, although the finite transverse variation in concentration formed an essential part of the mechanism. In the same way Stommel (1953) in using an effective K_x, regarded an estuary such as the Severn as being vertically homogeneous, although the inhomogeneity is again an essential part of the mixing process. At the other extreme, when the stability is so great that there is obviously a two-layer flow, with markedly different concentrations in the two layers, there is no advantage in treating the process as one of longitudinal diffusion.

As a rough criterion for estuaries and coastal waters, one might say that it is useful to consider the mixing as a process of longitudinal diffusion when the vertical difference in salinity, from surface to bottom, is less than about 1 ‰. In this case to determine the difference to within 10 % would require salinity measurements to an accuracy of ± 0·05 ‰, which would probably be of the same order as the random fluctuations in a region where the mixing of river water and sea water is taking place.

Another question is the relation of the shear effect to other processes of horizontal mixing. Unlike the processes envisaged in theories such as that of Joseph & Sendner, the shear effect does not involve a horizontal scale factor, but only the depth h of the water or of the layer in which the dispersion is taking place. It is implied that the horizontal scale of the process is at least an order of magnitude greater than h. In the examples given above, K_x was of the order $10^6\,\text{cm}^2/\text{sec}$ and according to Joseph & Sendner's theory (§ 5 above), with $P = 1$ cm/sec, this corresponds to $r = 10$ km. On a smaller scale the shear effect would be more important but for distances greater than 10 km it would seem that the random processes considered in the Joseph & Sendner theory would predominate. The fact that in Liverpool Bay the effective value of K_x was found to be lower than within the Mersey estuary indicates that in this case, which is probably typical of estuaries and coastal waters, the mechanism acting is different from that envisaged in the more general theories.

10. Conclusions

The following general conclusions are indicated by this investigation:

(1) Horizontal mixing may be produced by the shear effect in tidal currents, density currents or wind-driven currents.

(2) The mixing produced is comparatively small if the water is homogeneous, but the effect becomes important if a moderate degree of stability is present, corresponding to a local Richardson number of about 0·5 or 1. The coefficient of vertical eddy diffusion may then be reduced by a factor of 10 or 20, with a corresponding increase in the effective coefficient of horizontal diffusion. In these conditions the effective K_x and K_y may be of order 10^5 or 10^6 cm^2/sec.

(3) The shear effect can account for the diffusion being greater parallel to the main component of the current than in a direction perpendicular to it, and hence for the elongation of a patch of diffusing material.

(4) Diffusion by this process is likely to be most effective in estuaries and within a few miles of the coast. At greater distances from the coast, it is probable that the influence of other features, such as large horizontal eddies, will predominate.

The assistance of my colleagues in the Oceanography Department, University of Liverpool, in making the observations at sea is gratefully acknowledged.

REFERENCES

Bowden, K. F. 1963 *Int. J. Air. Wat. Poll.* **7**, 343–56.

Bowden, K. F. 1964 *Oceanography and Marine Biology: an Annual Review*, **2**, 11–30.

Bowden, K. F. & Fairbairn, L. A. 1952 *Proc. Roy. Soc.* A, **214**, 371–92.

Bowles, P., Burns, R. H., Hudswell, F. & Whipple, R. T. P. 1958 *Proc. 2nd Conf. Peaceful Uses of Atomic Energy, U.N. Geneva*, **18**, 376–89.

Cartwright, D. E. 1961 *J. Inst. Navig.* **14**, 130–51.

Csanady, G. T. 1963 *J. Fluid Mech.* **17**, 360–84.

Defant, A. 1961 *Physical Oceanography*, vol. 1. Oxford: Pergamon Press.

Ekman, V. W. 1905 *Ark. Mat. Astr. Fys.* **2** (11).

Elder, J. W. 1959 *J. Fluid Mech.* **5**, 544–60.

Ichiye, T. 1962 *J. Geophys. Res.* **67**, 3213–16.

Joseph, J. & Sendner, H. 1958 *Deut. Hydrog. Z.* **11**, 49–77.

Okubo, A. 1962 *J. Oceanogr. Soc. Japan*, 20th anniv. vol., pp. 286–320.

Stommel, H. 1953 *Sewage & Ind. Wastes*, **25**, 1065–71.

Taylor, G. I. 1953 *Proc. Roy. Soc.* A, **219**, 186–203.

Taylor, G. I. 1954 *Proc. Roy. Soc.* A, **223**, 446–468.

Van Veen, J. 1938 *J. Cons. int. Explor. Mer*, **13**, 7–36.